The study of SU(3) super Yang-Mills quantum mechanics.

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Abstract

We present the hamiltonian study of super Yang-Mills quantum mechanics (SYMQM). The recently introduced method based on Fock space representation allows to analyze SYMQM numerically. The detailed analysis for SYMQM in two dimensions for $SU(3)$ group is given.

1 Introduction

Supersymmetric Yang-Mills quantum mechanics are very interesting models since they emerge in different areas of physics. The general, not necessarily gauge, supersymmetric quantum mechanics have been studied first as a laboratory of supersymmetry [1] where in particular the exact solution for $D = 2$, $SU(2)$ case was given. By definition SYMQM are $\mathcal{N} = 1$ super Yang-Mills field quantum theories reduced from $D = d + 1$ to $D = 0 + 1$ dimensions. Supersymmetry requires the space-time dimension to be $D = 2, 4, 6, 10$ with $\mathcal{N} = 2, 4, 8, 16$ supercharges in the resulting quantum mechanics respectively. The rotational symmetry and gauge invariance of the original theory become now the internal Spin(d) and global $SU(N)$ symmetry. The physical states become now the $SU(N)$ singlets. We denote the spatial components of gauge field $A^i_a(t)$ by x^i_a and their conjugate momenta by p^i_a, $[x^i_a, p^j_b] = \delta^{ij}\delta_{ab}$. The hamiltonian is then [1]

$$H = \frac{1}{2}p^i_a p^i_a + \frac{1}{4}g^2(f_{abc}x^i_b x^j_c)^2 + H_F, \tag{1}$$

where $H_F = -\frac{i}{2}g f_{abc}\vartheta^\alpha_a x^i_b \Gamma^i_{\alpha\beta}\vartheta^\beta_c$ for $D = 2, 10$ and ϑ are real spinors obeying $\{\vartheta^\alpha_a, \vartheta^\beta_b\} = \delta^{\alpha\beta}\delta_{ab}$, $\alpha, \beta = 1, \ldots, \mathcal{N}$ or $H_F = ig f_{abc}\bar{\vartheta}^\alpha_a x^i_b \Gamma^i_{\alpha\beta}\vartheta^\beta_c$ for $D = 4, 6$ and ϑ are complex spinors obeying $\{\bar{\vartheta}^\alpha_a, \vartheta^\beta_b\} = \delta^{\alpha\beta}\delta_{ab}$ for $\alpha, \beta = 1, \ldots, \frac{\mathcal{N}}{2}$. The $\Gamma^i_{\alpha\beta}$ are matrix representation of an $SO(d)$ Clifford algebra $\{\Gamma^i, \Gamma^j\} = 2\delta^{ij}$.

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The growing interest in these models is due to the BFSS (Banks, Fischler, Shenker, Susskind) conjecture [4] where the $N \to \infty$ limit of Eg.(1) is argued to describes M-theory in the infinite momentum frame. This stimulated further work on asymptotic form of the ground state of D=9+1, SU(2), SYMQM [8] and the analysis of Witten index of (1). The index does not vanish only in D=10 where it is equal to 1 [9,10,11,12]. Despite the relevance to M-theory SYMQM have been studied earlier in different context. The bosonic part of (1) was discovered in pure Yang-Mills theory in the zero volume limit [2]. Later on it appeared as a regularization describing the quantum supermembrane [3]. The detailed study of the hamiltonian (1) shows that in bosonic sector the potential is confining and there is no continuous spectrum [6]. If however the supersymmetry is turned on then there are bound states in fermion rich sectors as well as scattering ones [7].

The only exact solutions of (1) existing in the literature are for D=1+1, $SU(2)$ [1] and its generalization for arbitrary $SU(N)$ [5]. Therefore any numerical approach is of interest.

The plan of this paper is the following. In section 2 we briefly outline the method used to study the models just described and quote existing results in D=1+1,3+1,9+1 for $SU(2)$ group. In section 3 and 4 we study general properties in D=1+1 for arbitrary $SU(N)$ and present the results in D=1+1, $SU(3)$.

2 Cutoff method

The cutoff method [13] consists of numerical analysis of the hamiltonian in the occupation number representation. First we introduce the bosonic and fermionic creation and annihilation operators ${a^\dagger}^i_a$, a^i_a, ${f^\dagger}^\alpha_a$, f^α_a i.e.

$$a^i_a = \frac{1}{\sqrt{2}}(x^i_a + ip^i_a), \qquad [a^i_a, {a^\dagger}^j_b] = \delta^{ij}\delta_{ab}, \qquad \{f^\alpha_a, {f^\dagger}^\beta_b\} = \delta^{\alpha\beta}\delta_{ab}{}^1.$$

Next we truncate the Hilbert space to the maximal number o quanta

$$n_B = \sum_{i,b} {a^\dagger}^i_b a^i_b, \qquad n_B \le n_{Bmax},$$

compute matrix elements of H and diagonalize the resulting finite matrix. In this way one can analyze the spectrum dependence on a cutoff n_{Bmax}. There is a dramatic difference between the behavior of the continuous and discrete spectrum with cutoff. Namely

$$E^{n_{Bmax}}_m = E_m + O(e^{-n_{Bmax}}) \quad - \quad \text{discrete spectrum},$$

$$E^{n_{Bmax}}_m = O(\frac{1}{n_{Bmax}}) \quad - \quad \text{continuous spectrum},$$

where m is an index of the energy level $m = 1, \ldots, n_{Bmax} + 1$. The limit $n_{Bmax} \longrightarrow \infty$ is called the continuum limit. In the case of the discrete spectrum case the energy levels converge rapidly to the exact eigenvalues of the hamiltonian. This may not be surprising, however it is interesting

[1] There are several choices of fermionic f^α_a, ${f^\dagger}^\alpha_a$ operators. Since we do not make any explicit calculations here we refer the reader to [13] for details.

to see how fast is the convergence. For details the reader is referred to [14]. In the continuous spectrum case things are different. The convergence is very slow and all the eigenvalues vanish in the infinite cutoff limit. In the continuum limit the spectrum is continuous and the only way to restore it from cut Fock space is to put the following scaling [15]

$$m(n_{Bmax}) = const.\sqrt{n_{Bmax}} \Longleftrightarrow E^{n_{Bmax}}_{m(n_{Bmax})} \to E. \tag{2}$$

It was claimed in [15] that this scaling law should work independently of the theory whenever one can define scattering states asymptotically. The argument for the above claim is based on the following fact. The eigenvalues of the momentum operator in ordinary d=1 quantum mechanics in cut Fock space are zeros of Hermite polynomials $H_{n_{Bmax}}(x)$ the asymptotic behavior of which is $\frac{1}{\sqrt{n_{Bmax}}}$ [14,15]. Therefore, once the momentum operator is defined, its spectrum cutoff dependence should be $\frac{1}{\sqrt{n_{Bmax}}}$ for large n_{Bmax}.

The $E^{n_{Bmax}}_m$ values for fixed n_{Bmax} give the opportunity to calculate regularized (n_{Bmax} dependent) Witten index. If the spectrum of the supersymmetric hamiltonian H is discrete then the index counts the difference between bosonic n^0_b and fermionic n^0_f ground states i.e.

$$I_W = Tr(-1)^F e^{-\beta H} = \sum_m (-1)^{F(m)} e^{-\beta E_m} = n^0_b - n^0_f,$$

where F is a fermion number. This quantity is β independent. The cutoff makes it β and n_{Bmax} dependent i.e.

$$I^{reg}_W(\beta, n_{Bmax}) = \sum_{m=1}^{n_{Bmax}+1} (-1)^{F(m)} e^{-\beta E^{n_{Bmax}}_m}. \tag{3}$$

If the spectrum of the hamiltonian H is continuous then the I_W depends on β and the difference $n^0_b - n^0_f$ may be obtained by taking the $\beta \to \infty$ limit. On the other hand the $\beta \to 0$ limit is easier to compute, therefore one introduces the boundary term δI_W using the following trick [9]

$$\delta I_W = I_W(\infty) - I_W(0) = \int_0^\infty d\beta \frac{d}{d\beta} I_W(\beta).$$

2.1 D=1+1,3+1,9+1 SU(2) SYMQM

In $D = 1+1$ case the hamiltonian $H = \frac{1}{2} p_a p_a + g x_a G_a$, where G_a is the SU(N) generator, is free in a gauge invariant sector. There are as many fermion sectors as the grassmann algebra allows i.e. 1 boson sector and $N^2 - 1$ fermionic sectors. Since the gauge group is $SU(2)$ we will denote them as $\mid F = 0\rangle$, $\mid F = 1\rangle$, $\mid F = 2\rangle$, $\mid F = 3\rangle$. We also have the particle-hole symmetry which relates sectors $\mid 0\rangle \leftrightarrow \mid 3\rangle$ and $\mid 1\rangle \leftrightarrow \mid 2\rangle$ hance the analysis of the first two sectors is sufficient. There is also supersymmetry which relates sectors $\mid 0\rangle \leftrightarrow \mid 1\rangle$ and $\mid 2\rangle \leftrightarrow \mid 3\rangle$ therefore the whole information about the spectrum is in fact in the first sector. Supersymmetry does not communicate between sectors $\mid 1\rangle$ and $\mid 2\rangle$ which is exceptional for $SU(2)$. Since the particle-hole symmetry relates sectors with different fermion number, it is evident that the regularized Witten index of this model vanishes. It is however interesting to compute the restricted Witten index

which is defined in first two sectors only and the exact answer is $\frac{1}{2}$ [16] which was also confirmed numerically.

In D=3+1 dimensions the hamiltonian (1) is not free due to the quartic potential term. There are 6 fermionic sectors. The particle-hole symmetry relates sectors with the same fermion number hance the eigenstates from these sectors do not cancel under the sum (3). The analysis of the index [10] shows that in this case

$$I_W(\infty) = I_W(0) + \delta I = \frac{1}{4} - \frac{1}{4} = 0 \quad \text{Witten index for D=3+1, SU(2).}$$

On the contrary the cutoff analysis gives the non zero value [17]. The index converges towards $\frac{1}{4}$ which is exactly the value of the $I_W(0)$ not $I_W(\infty)$. It seems that the cutoff method somehow does not contain the boundary term δI.

This model is the first non-trivial one where the scaling (2) was confirmed i.e. the spectrum of a free particle $p^2/2$ can be recovered provided eqn. (2) is applied. Moreover, in fermion rich sectors both discrete and continuous spectrum is present which precisely corresponds to conclusions of [7].

The analysis of the supermultiplets is even more interesting. Each eigenstate is labelled by three quantum numbers: energy E, angular momentum l and fermion number F. Therefore each state can be represented by a dot in R^3 space. It can be proved [17] that supersymmetry links these dots in such a way that the emerging geometrical object representing each supermultiplet is a diamond. This picture very nicely catalogues all the supermultiplets and it is independent of a gauge group.

In $D = 9+1$ dimensions case we only note the astonishing difficulties that emerge [18]. Since we have the SO(9) symmetry the second order Casimir operator is

$$J^2 = \sum_{i<k} J_{ik}, \qquad J_{ik} = x_a^{[i} p_a^{k]} + \frac{1}{2}\psi_a^\dagger \Sigma^{ik} \psi_a, \qquad \Sigma^{ik} = -\frac{i}{4}[\Gamma^i, \Gamma^k].$$

Normally we would have expect the SO(9) singlet to be the Fock vacuum $\mid 0\rangle$. This is not he case here since one can prove that $J^2 \mid 0\rangle = 78 \mid 0\rangle$ [18]. The empty state is not invariant under rotations! This is a surprising fact and it means that the SO(9) singlet is somewhere else. Where is it? The model has 24 fermionic sectors and it was found that the singlet happens to be just in the central F=12 sector.

3 The general properties ot the D=1+1, SU(N) SYMQM

Since the eigenstates in SYMQM are the gauge singlets therefore it is reasonable to ask about the convenient $SU(N)$ invariant basis. It is evident that states belonging to such basis have to be of the form

$$T_{bc\ldots de\ldots} a_b^\dagger a_c^\dagger \ldots f_b^\dagger f_c^\dagger \ldots \mid 0\rangle, \tag{4}$$

where $T_{bc\ldots de\ldots}$ is some SU(N) invariant tensor made out of structure tensors $f_{abc}, d_{abc}, \delta_{ab}$. We now proceed to chose linearly independent states from (4).

3.1 Birdtracs

In order to deal with the variety of all possible tensor contractions we introduce the diagrammatic approach (figure 1).

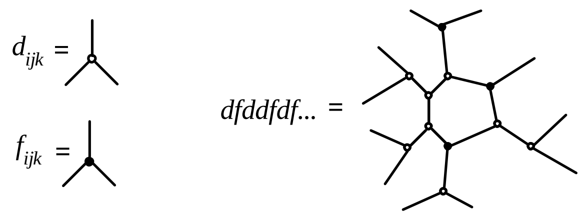

Figure 1: Diagrammatic notation of invariant tensors.

Each leg corresponds to one index and summing over any two indices is simply gluing appropriate legs. Structure tensors f_{ijk}, d_{ijk} are represented by vertices and δ_{ij} is a line. Any tensor may now be represented by a graph. Such diagrammatic approach has already been introduced long time ago by Cvitanovič [19]. In, general one can construct loop tensor which by definition is a tensor that diagrammatically looks like a loop however it can be proved [20] that any such loop can be expressed in terms of forests i.e. products of tree tensors (figure 2)

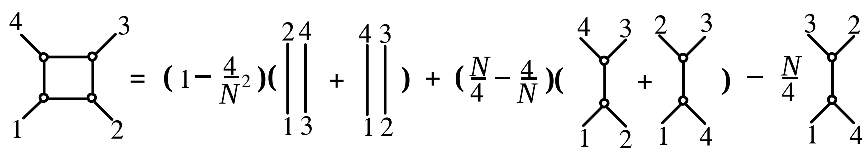

Figure 2: An example of loop reduction for a square made out of d_{ijk} tensors.

Therefore we are left with tree tensors only. These however can be easily expressed in terms of trace tensors $Tr(T_a T_b \ldots)$ where T_a are $SU(N)$ generators in fundamental representation. With the use of the following matrices $A^\dagger = a_b^\dagger T_b$, $F^\dagger = f_b^\dagger T_b$ any gauge invariant state can be obtained by acting with an appropriate linear combination of products of trace operators

$$Tr({A^\dagger}^{i_1} F^\dagger {A^\dagger}^{i_2} F^\dagger \ldots {A^\dagger}^{i_k} F^\dagger),$$

on Fock vacuum $\mid 0\rangle$. Due to the grassmann algebra the number of F matrices under the trace cannot be grater then N^2-1 i.e. $k \leq N^2-1$. Moreover the Cayley-Hamilton theorem for A matrices gives $i_k \leq N$. The remaining set of states is still linearly dependent and the further analysis requires separate study of each $SU(N)$. The basis states in F=0 sector are of the form

$$\mid i_2, i_3, \ldots, i_N\rangle = Tr^{i_2}({A^\dagger}^2) Tr^{i_3}({A^\dagger}^3) \ldots Tr^{i_N}({A^\dagger}^N) \mid 0\rangle.$$

We see that there are as many states with given number of quanta n_B as there are natural solutions of the equation $2i_2 + 3i_3 + \ldots + N i_N = n_B$. For $U(N)$ this would be exactly $p(n_B)$

- the partition number of n_B. For $SU(N)$ this is a little less then $p(n_B)$ however it still grows exponentially with n_B.

In order to solve the model in bosonic sector one has to compute the following scalar product

$$N^{i_2\ldots i_N}_{j_2\ldots j_N} = \langle i_2 \ldots i_N \mid j_2 \ldots j_N\rangle,$$

which in principle is a tedious, but not impossible, task .

Let us discuss, the "bilinear" basis which by definition is the following restricted $SU(N)$ basis

$$\mid 2n\rangle = (A^\dagger A^\dagger)^n \mid 0\rangle, \qquad (A^\dagger A^\dagger) = a_i^\dagger a_i^\dagger \tag{5}$$

which was introduced in [2] in D=3+1 case. In this basis the non zero hamiltonian matrix elements are easy to derive. First we write the commutation relations

$$[(AA),(A^\dagger A^\dagger)^n] = 4n(A^\dagger A^\dagger)^{n-1}(A^\dagger A) + 4n(n-1+\frac{N^2-1}{2})(A^\dagger A^\dagger)^{n-1}, \tag{6}$$

$$[(AA^\dagger),(A^\dagger A^\dagger)^n] = 2n(A^\dagger A^\dagger)^n. \tag{7}$$

Using (6) we obtain norms for $\mid 2n\rangle$ i.e.

$$c_{2n}^2 := \langle 2n \mid 2n\rangle = 4n(n-1+\frac{N^2-1}{2})c_{n-1}^2, \qquad c_{2n} = \sqrt{\prod_{k=1}^{n} 4k(k-1+\frac{N^2-1}{2})}, \qquad c_0 = 1.$$

In the orthonormalized basis $\mid \tilde{2n}\rangle = \frac{1}{c_{2n}} \mid 2n\rangle$ the non vanishing matrix elements of the hamiltonian

$$H = \frac{1}{2}p_a p_a = -\frac{1}{4}((A^\dagger A^\dagger) + (AA) - 2(A^\dagger A) - (N^2-1)),$$

are

$$\langle \tilde{2n} \,|H|\, \tilde{2n}\rangle = n + \frac{N^2-1}{4}$$

and

$$\langle \widetilde{2n+2} \,|H|\, \tilde{2n}\rangle = \langle \tilde{2n} \,|H|\, \widetilde{2n+2}\rangle = -\frac{1}{2}\sqrt{(n+1)(n+N^2-1)}.$$

Therefore it is straightforward to proceed with the cutoff analysis (figure 3). We see that thee is no quantitative difference between $SU(2)$ and eg. $SU(100)$ case. This is not what we have expected and it means that the restricted basis (5) simplifies too much.

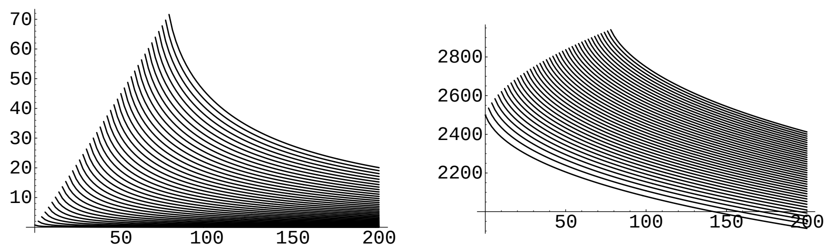

Figure 3: The cutoff dependence of spectrum for $SU(2)$ and $SU(100)$ in "bilinear" basis.

4 D=1+1, SU(3) SYMQM

Here we present the calculations of Hamiltonian matrix elements in a complete basis in bosonic sector. The basis vectors and the scalar products, we are interested, in are

$$\mid i,j\rangle = (A^\dagger A^\dagger)^i (A^\dagger A^\dagger A^\dagger)^j \mid 0\rangle, \qquad N^{i\ j}_{i'\ j'} = \langle i,j \mid i',j'\rangle. \tag{8}$$
$$(A^\dagger A^\dagger A^\dagger) = d_{ijk} a^\dagger_i a^\dagger_j a^\dagger_k$$

The only non vanishing elements of $S^{i\ j}_{i'\ j'}$ are the ones obeying the constraint $2i+3j = 2i'+3j'$. Therefore it is convenient to work with the following symbol

$$W^k_{i\ j} = \langle i,j \mid (AAA)^{2k}(A^\dagger A^\dagger)^{3k} \mid i,j\rangle,$$

which has the advantage of reproducing all non vanishing $N^{i\ j}_{i'\ j'}$'s. It is tedious but possible to obtain formulas and recurrence equations for W^k_{ij}. We shall omit the lengthy derivation and only give the results.

First we solve the recurrences for W^k_{00} and W^k_{i0}. We have

$$W^k_{00} = 96k(2k-1)(9k^2-1)(9k^2-4)W^{k-1}_{00}, \quad W^k_{i0} = 4(3k+i)(3k+i+3)W^k_{i-1\ 0}, \quad W^0_{00} = 1. \tag{9}$$

therefore (9) gives an exact formula for W^k_{i0}. The W^k_{0j} term is computed from the following recurrence

$$W^k_{0j} = \alpha_{jk} W^{k-1}_{0j} + \beta_{jk} W^k_{0j-2} + \gamma_{jk} W^{k+1}_{0j-4},$$

where

$$\alpha_{jk} = 48(2k+j)(2k+j-1)(3k-1)(3k-2)(3k+3j+2)(3k+3j+1),$$
$$\beta_{jk} = 72(2k+j)(2k+j-1)j(j-1)(9k^2+9kj-2),$$
$$\gamma_{jk} = 27(2k+j)(2k+j-1)j(j-1)(j-2)(j-3).$$

This recurrence stops on W^k_{0j} given by (9). The general term W^k_{ij} is now computed from yet another recurrence

$$W^k_{ij} = 4(i+3k)(i+3k+3j+3)W^k_{i-1j} + 3j(j-1)W^{k+1}_{i-1j-2},$$

which stops on W^k_{0j} and W^k_{i0}. The whole norm matrix (8) can now be computed. It should be noted that the elements of the N matrix were obtained independently by computing the scalar products $\langle i,j \mid i',j' \rangle$ with use of the program written in Mathematica [13]. In this way all the recurrences presented here were confirmed up to $n_B = 12$ i.e. for (i,j) such that $2i+3j \le 12$. This matrix is in fact the Gram matrix which indicates that we still have to orthogonalize the basis. We will not do so however. In order to represent the hamiltonian H in orthogonal basis we follow [21]. It is sufficient to calculate the Gram matrix G and proceed with the following similarity transformation

$$H_{ort} = G^{-\frac{1}{2}} H G^{-\frac{1}{2}}.$$

The results of the cutoff analysis are presented in figure 4.

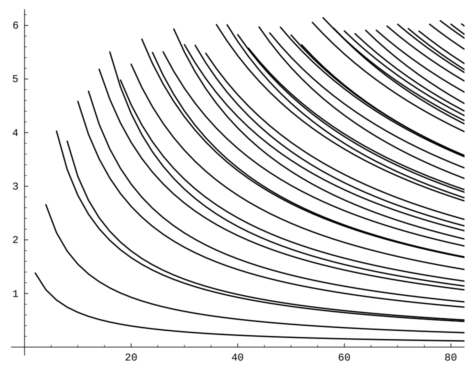

Figure 4: The cutoff dependence of spectrum in $D = 1+1$, $SU(3)$, $F = 0$.

It is clear that the spectrum seems to be far more complicated than in $SU(2)$ case. The lines in figure 4 are divided into groups where they converge together. This can be understood in the following way. In $SU(3)$ we have two Casimir operators $T_a T_a$ and $d_{abc} T_a T_b T_c$ where T_a's are $SU(3)$ generators. In cut Fock space the second one does not commute with the hamiltonian therefore the cutoff n_B breaks the $SU(3)$ symmetry. In $n_B \to \infty$ limit the symmetry should be restored which corresponds to grouping of the lines in figure 4.

5 Summary

SYMQM models reveal variety of application in several areas of physics (Yang-Mills theories, supersymmetry, strings) hance their detailed analysis is of interest. Although they are rich in symmetries ($SU(N)$, SO(d), supersymmetry) the exact solutions are missing in the literature forcing one to apply numerical methods. The cutoff method presented here is working surprisingly well, however to get any of results of sections [2,3,4] one had to employ a lot of theoretical work which in some cases gave exact results (e.g. the structure of supermultiplets). The analysis of D=1+1 SYMQM for arbitrary $SU(N)$ is very encouraging and gives a hope to proceed with the $N \to \infty$ limit.

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